IMPORTANT COVARIANT CURVES AND A COMPLETE STSTEM OF INVARIANTS OF THE RATIONAL QUARTIC CURVE*

BY

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Introduction.

It is well understood that different domains of rationality are useful in discussing different properties of curves. Two domains are employed in the following, to render possible the geometric interpretation of certain invariants and covariant loci of the rational plane quartic.

Section 1 is divided into two parts: In the first part is given a straightforward proof of the covariance of curves derived from R^n by a certain translation scheme; in the second part SALMON's work on the combinants of two binary quartics is applied to those covariant curves of the R^4 which can be found as combinants. In Section 2 the most important invariants of the R^4 are discussed, and four invariants are found in terms of which any other invariant relation on the R^4 can be expressed algebraically; in this sense these four invariants constitute an *algebraically complete system.* Section 3 contains a treatment of the invariants of the R^4 when the R^4 is taken as the section of the Steiner Quartic Surface by a plane; in this scheme the invariants occur as symmetric functions of the coefficients of the cutting plane.

§ 1*a*. *Certain Covariants of Rational Curves.*

Let the rational curve of order n, which may be called R^n, be written parametrically

(1) $$x_i = a_i t^n + b_i t^{n-1} + c_i t^{n-2} + \cdots \qquad (i = 0, 1, 2).$$

If (1) is cut by two lines

(2) $$(\xi x) = \xi_0 x_0 + \xi_1 x_1 + \xi_2 x_2 = 0,$$

and

(3) $$(\eta x) = \eta_0 x_0 + \eta_1 x_1 + \eta_2 x_2 = 0,$$

the results of this operation are the two binary n-ics

(4) $$(a\xi) t^n + (b\xi) t^{n-1} + (c\xi) t^{n-2} \cdots = 0,$$

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and

$$(5) \qquad (a\eta)t^n + (b\eta)t^{n-1} + (c\eta)t^{n-2} \cdots = 0,$$

which yield the n parameters of the points in which the lines (2) and (3) cut the curve.

The combinants of (4) and (5) are expressible as rational functions of the two-rowed determinants of the matrix

$$(6) \qquad \left\| \begin{matrix} (a\xi) & (b\xi) & (c\xi) & (d\xi)\cdots \\ (a\eta) & (b\eta) & (c\eta) & (d\eta)\cdots \end{matrix} \right\|.$$

But if in any two-rowed determinant of (6), the quantities x_0, x_1, x_2 are substituted for the coordinates of the point in which (2) and (3) intersect, the result may be expressed as a three-rowed determinant. For instance,

$$(7) \qquad \begin{vmatrix} (a\xi) & (b\xi) \\ (a\eta) & (b\eta) \end{vmatrix} = \begin{vmatrix} a_0 & b_0 & x_0 \\ a_1 & b_1 & x_1 \\ a_2 & b_2 & x_2 \end{vmatrix} = |\, abx \,|;$$

and evidently the other determinants of (6) assume similar form. Consequently, by means of the above translation scheme * from the combinants of (4) and (5) certain loci are derived which are related to the R^n in a special manner. Certain sets of points on the R^n defined by some projective relation are connected in this way with a point of the plane, namely, the intersection of (2) and (3). This fact alone warrants the assertion that loci related to the R^n in such a manner are *covariant curves* † of the R^n. But it is important to give an actual algebraic proof of this fact.

In order to prove formally that these curves are covariants, it is sufficient to show that they are unaltered by any linear transformation of the x's or t's.

Let the transformation of the x's be

$$(8) \qquad x_i' = l_i x_0 + m_i x_1 + n_i x_2 \qquad (i = 0, 1, 2).$$

After this transformation (1) assumes the form

$$(9) \qquad x_i' = (l_i a_0 + m_i a_1 + n_i a_2)t^n + (l_i b_0 + m_i b_1 + n_i b_2)t^{n-1} \cdots,$$

or

$$(10) \qquad x_i = (l_i a)t^n + (l_i b)t^{n-1} + (l_i c)t^{n-2} \cdots,$$

where $(l_i a) = l_i a_0 + m_i a_1 + n_i a_2$, etc.

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* GRACE and YOUNG, *Algebra of Invariants*, pp. 314–318.

† GRACE and YOUNG in the place already referred to have practically proved that such loci are covariant.

From what has been stated the combinant curves of (10) are expressible as sums of terms, each term being of the same degree in determinants of the type

$$\text{(11)} \qquad \begin{vmatrix} (l_0 a) & (l_0 b) & (l_0 x) \\ (l_1 a) & (l_1 b) & (l_1 x) \\ (l_2 a) & (l_2 b) & (l_2 x) \end{vmatrix}.$$

But by the theorem for the multiplication of determinants (11) becomes

$$\text{(12)} \qquad |lmn|\,|abx|;$$

also each three-rowed determinant of the type (11) formed from (10) is equal to the product of $|lmn|$ by the corresponding determinant formed from (1). Hence, as $|lmn|$ factors out of each term of the combinants of (10) to the same degree, these curves are unaltered by a transformation of the x's. This is a property which is possessed by any homogeneous function of the determinants of the type $|abx|$; however, not every such function is a covariant. But if such a function is also unaltered by a linear fractional transformation of the parameter t, then such a function is a covariant. Therefore it remains to show that combinant curves possess also this property.

Let the transformation of the t's $(t = t_1/t_2)$ be

$$\text{(12)} \qquad t_1 = m_1 t_1 + m_2 t_2, \qquad t_2 = l_1 t_1 + l_2 t_2.$$

Transforming (1) by means of (12) we obtain

$$\text{(13)} \qquad x_i = a_i' t^n + b_i' t^{n-1} + c_i' t^{n-2} \cdots \qquad (i = 0, 1, 2).$$

If we cut (13) by the two lines (2) and (3) we shall obtain two binary n-ics,

$$\text{(14)} \qquad (a'\xi)t^n + (b'\xi)t^{n-1} \cdots = 0,$$

and

$$\text{(15)} \qquad (a'\eta)t^n + (b'\eta)t^{n-1} \cdots = 0.$$

But (14) and (15) are exactly what we should have obtained by operating upon (4) and (5) with (12). However, the combinants of (4) and (5) are unaltered by any such transformation as (12); hence *the combinant curves are unaltered by a transformation of the parameter t and are covariant curves of the R^n.* All loci obtained in the manner indicated are covariants, but other covariants occur which cannot be derived as combinants.

Salmon * has discussed the combinants of two binary quartics from an algebraic standpoint and we shall apply his result with some extensions to a study of those covariants of the R^4 which can be derived in this manner.

* *Higher Algebra*, third edition, pp. 200–206.

§ 1b. *Covariant Curves of R^4 as Combinants.*

Let the equation of the R^4 be written parametrically

$$x_i = a_i t^4 + 4b_i t^3 + 6c_i t^2 + 4d_i t + e_i \qquad (i=0,1,2). \tag{16}$$

If the R^4 is cut by the two lines (2) and (3) we obtain the two binary quartics,

$$U = (a\xi)t^4 + 4(b\xi)t^3 + 6(c\xi)t^2 + 4(d\xi)t + (e\xi) = 0, \tag{17}$$

$$V = (a\eta)t^4 + 4(b\eta)t^3 + 6(c\eta)t^2 + 4(d\eta)t + (e\eta) = 0, \tag{18}$$

giving the parameters of the points in which (2) and (3), respectively, cut the curve. Considering the u and v above to be the same as those of Salmon * by reason of (5) and (7) we have

$$\begin{gathered}\alpha=(ab)=|abx|,\quad \alpha'=(de)=|dex|,\quad \beta=(ad')=|adx|,\quad \beta'=(be')=|bex|,\\ \gamma=(ae')=|aex|,\quad \delta=(bd')=|bdx|,\quad \lambda=(ac')=|acx|,\quad \lambda'=(ce')=|cex|,\\ u=(be')=|bcx|,\quad u'=(cd')=|cdx|.\end{gathered} \tag{19}$$

Consider the pencil $u + Kv$ as a single binary quartic, substitute its coefficients in its two invariants g_2 and g_3, and let the resulting quadratic and cubic in K be written

$$S_0'K^2 + S_1'K + S_2' = 0 \quad \text{and} \quad T_0K^3 + T_1K^2 + T_2K + T_3 = 0; \tag{20}$$

the quadratic gives those two values of K for which $u + Kv$ becomes a self-apolar quartic; and the cubic those values of K for which $u + Kv$ becomes a catalectic quartic.

Equating to zero the discriminant of the quadratic of (20) and making the substitutions (7), we have

$$\begin{aligned}A \equiv |aex|^2 + 16|bdx|^2 + 12|acx||cex| - 48|bex||cdx| - 8|abx||dex|\\ - 8|adx||bex| = 0,\end{aligned} \tag{21}$$

which is the locus whose tangents cut R^4 in self-apolar sets of points. If the g_2 of (4) were formed it would be the envelope of lines which cut R^4 in self-apolar quartics. Hence A is the point equation of the conic g_2. A line which cuts R^4 in three consecutive points is a line of A, for the g_2 of a quartic having a triple root vanishes. Consequently the six inflexional tangents of R^4 touch A, and indeed A is often spoken of as the conic on the six inflexional tangents of R^4.

The condition for u and v to have the same apolar cubic, or that each may be expressed linearly in the same three fourth powers, leads to a second combinant which with the substitutions (7) becomes

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* *Higher Algebra*, third edition, pp. 200–206.

$$(22)\quad \begin{aligned} B \equiv |acx||cex| - |bcx||cdx| - |adx||cdx| \\ - |bex||bcx| + |bdx|^2 - |abx||dex| = 0. \end{aligned}$$

This conic may be identified with the conic which Stahl* gives parametrically; it is the locus of the vertices of flex-triangles of all first osculants.

The eliminant of u and v expresses the condition for a common root or the condition that they intersect on the curve. Hence by reason of (7) R becomes the point equation of the R^4.

The Jacobian of u and v is

$$(23)\quad \alpha t^6 + 3\lambda t^5 + (3\beta + 6\mu)t^4 + (\gamma + 8\delta)t^3 + (3\beta' + 6\mu')t^2 + 3\lambda' t + \alpha' = 0.$$

By a well known property of the Jacobian (23) gives the six tangents from a point to R^4. The condition that these form a self-apolar set is

$$(24)\quad -40 I_2 \dagger = A + 48B.$$

Consequently the locus of points such that tangents from them to R^4 form self-apolar sets is the conic

$$(25)\quad A + 48B = 0.$$

Further we find that the following relation holds

$$(26)\quad 320(I_2^2 - 100 I_4) = (A - 16B)^2 - R,$$

where I_2 and I_4 are the well known invariants of the sextic (23). But $I_2^2 - 100 I_4 = 0$ for a sextic which has either a triple root, or a double root and the other four forming a self-apolar set. Consequently such points lie on the quartic curve

$$(27)\quad (A - 16B)^2 - R = 0.$$

The points on R which behave in this manner are on the conic $A - 16B = 0$. Hence the six flexes and the two points q_i (which have the property that tangents drawn from them to R^4 form self-apolar sets) lie on the conic

$$(28)\quad A - 16B = 0.$$

It is worthy of notice that the above not only gives an easy way to write down the equation of the conic on the flexes but is an independent analytical proof that they do lie on a conic with the two points q_i.

Salmon's combinant equation $C = 0$, expressing the condition that a member of the pencil $u + Kv$ can have two squared factors, and from our standpoint the locus of points such that lines drawn from them cut the R^4 in quartics which

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* W. STAHL, Journal für die reine und angewandte Mathematik, vol. 101 (1886).

† $I_2 = 0$ is the self-apolarity condition of (23).

have two squared factors, can be nothing else than the point equation of the four double tangents of the R^4. Since it may be written

$$(29) \qquad 128\,C = (A - 48B)^2 - R,$$

it is evident that the conic on the eight points of contact of the four double tangents is

$$(30) \qquad A - 48B = 0.$$

Forming the Hessian of the cubic of (20) and requiring it to be apolar to the quadratic of (29) yields a combinant which may be expressed thus,

$$(31) \qquad 128I = R - (A - 16B)(A + 48B).$$

When $I \equiv 0$, the R^4 breaks up into two conics. If the two lines from a point which cut the R^4 self-apolarly are apolar to the Hessian pair of the three lines which cut R^4 in catalectic quartics, then I vanishes; therefore together with (7) it gives the locus of such points.

Forming the third transvectant of u and v we get a quadratic in t whose discriminant vanishes on the locus

$$(32) \qquad A - 12B = 0.$$

This may be identified with Stahl's conic N. The first osculant of R^4 at a point t' is an R^3 and therefore has three flexes on a line. The equation of this flex-line is a quadratic in t' and the discriminant of this quadratic may be identified with (32). Or writing the R^4 symbolically

$$(33) \qquad (a\xi)(\alpha t)^4 = 0 \qquad \text{and} \qquad (b\xi)(\beta t)^4 = 0,$$

then taking the third transvectant we have

$$(34) \qquad |abx||\alpha\beta|^3(\alpha t)(\beta t) = 0,$$

which is the conic N given parametrically. Of course the letters in (33), (34) are only symbols and must not be confused with the same letters already used with different meanings.

The eliminant of the cubic and quadratic of (20) is the condition for such a K_1 to exist that $u + K_1 v = 0$ has roots that are both self-apolar and harmonic pairs. But a quadratic for which $g_2 = g_3 = 0$ has a triple root, and therefore with (7) Salmon's $D = 0$ becomes the equation of the six flex tangents of the R^4.

The combinant E, the discriminant of the cubic of (20), evidently yields the locus of point whose lines cut R^4 in catalectic quartics. It is therefore the point equation of the g_3 of equation (4).

Suppose that the polar of the quadratic of (20) as to the cubic of (20) is ϕ_1, and the polar of ϕ_1 as to the quadratic is ϕ_2, by using the cubicovariant of the

cubic instead of the cubic itself we obtain ϕ_1' and ϕ_2' by the same operations. The condition for $\phi_1 = 0$ and ϕ_2' (or ϕ_1' and ϕ_2) to be the same is the combinant M, and by using (7) we have the locus of such points.

Before leaving covariant curves we observe that from Salmon's identities

$$D - E = -2IB - B^2(A - 16B), \tag{35}$$

and

$$128I = R - (A + 48B)(A - 16B), \tag{36}$$

it follows that

$$64(D - E) = B[(A - 16B)^2 - R]. \tag{37}$$

Consequently,* *twelve of the intersections of the flex tangents of R^4 and E lie on the conic B.*

Also from (26) and (37) we have

$$320(D - E) = B[(A + 48B)^2 - 100I_4]. \tag{38}$$

If $I_4 \equiv 0$ (i. e. if tangents drawn from any point to R^4 form a catalectic set), the intersections of D and E lie on two conics; the flex tangents touch E along $A + 48B = 0$ and intersect E along B. This particular R^4 is of the lemniscate type.

§ 2. *Invariants of R^4 as Combinants of the Fundamental Involution.*

If the R^4 be referred to a special triangle of reference its equations are

$$\begin{aligned} x_0 &= at^4 + 4bt^3, \\ x_2 &= 4dt + e, \\ x_2 &= 4bt^3 + 6ct^2 + 4dt. \end{aligned} \tag{1}$$

The condition for R^4 to have a triple point,† or the condition for R^4 to have a perspective point for (1) becomes

(2) $B_1' \equiv 36a^2c^2e^2 + 256ab^2d^2e - 96a^2cd^2e - 96ab^2ce^2 - 16a^2bde^2 = 0$.

If the g_2 of u of Section 1 is found for (1), its discriminant multiplied by eight may be written

(3) $A_1' = -6a^2c^2e^2 - 96b^2c^2d^2 + 8a^2bde^2 + 48abc^2de - 32ab^2d^2e = 0$.

Since the six inflexional tangents are lines of the conic g_2, A' vanishes when the R^4 has three concurrent flex tangents.

*See THOMSEN, American Journal of Mathematics, vol. 32 (1910), p. 222.

† W. STAHL, Mathematische Annalen, vol. 38 (1891); GROSS, Mathematische Annalen, vol. 32 (1888).

Calculating the conic on the flexes from the preceding section for (1) and taking its discriminant, we have

$$
\begin{aligned}
C_1' \equiv\ & 256\,(a^3b^2cd^4e^2)^* - 64\,(a^4c^2d^4e^2) + 32\,(a^4c^3d^2e^3) \\
(4)\qquad & - 192a^3b^2c^2d^2e^3 - 512a^2b^3c^2d^3e^2 + 128\,(a^3bc^3d^3e^2) \\
& - 32a^3bc^4de^3 - 4a^4c^4e^4 = 0.
\end{aligned}
$$

Since it is known that if an R^4 has three flexes on a line it has a fourth, $C' = 0$ is the condition for four collinear flexes of the R^4.

These three invariants are independent, as may be shown in the following manner: A' and B' are evidently independent. Further if C' were expressible in terms of A' and B', such a relation as $C' = \lambda B'^2 + \mu A'B'$ would have to hold because C' contains no term in $b^4c^4d^4$, i. e., B' would have to be a factor of C', but this is disproved by making $b = 0$ in (2), (3) and (4). Similarly D' is shown to be independent of A', B', and C'.

The condition for $A - 16B$ in lines to be apolar to A in points we shall call $R_2' = 0$, and we find that

$$
(5)\qquad 36R_2' + 108C_1' + B_1'^2 + 10A_1'B_1' + 16A_1'^2 = 0.
$$

If $H = 0$ is the apolarity condition of A in lines to $A - 16B$ in points, then

$$
(6)\qquad 6H + A_1'B_1' + 8A_1'^2 = 0.
$$

The discriminants of A and B are known, and therefore, the discriminant of the whole pencil

$$
(7)\qquad \lambda(A - 16B) + \mu A = 0
$$

is

$$
(8)\qquad C_1'\lambda^3 + R_2'\lambda^2\mu + H\lambda\mu^2 - A_1'^2\mu^3 = 0.
$$

But a more convenient formula than (8) is the discriminant of

$$
(9)\qquad \lambda'(cA + dB) + \mu'(aA + 16bB) = 0,
$$

which is

$$
\begin{aligned}
& \left[-(c+d)^3A_1'^2 - d(c+d)^2H + d^2(c+d)R_2' - d^3C_1'\right]\lambda'^3 \\
& + \Big[-3(a+b)(c+d)^2A_1'^2 - \big(b(c+d)^2 + 2d(a+b)(c+d)\big)H \\
& \qquad + \big(d^2(a+b) + 2bd(c+d)\big)R_2' - 3bd^2C_1'\Big]\lambda'^2\mu' \\
(10)\qquad & + \Big[-3(a+b)^2(c+d)A_1'^2 - \big(2(a+b)(c+d)b + (a+b)^2d\big]H \\
& \qquad + \big(2bd(a+b) + (c+d)b^2\big)R_2' - 3b^2dC_1'\Big]\lambda'\mu'^2 \\
& + \left[-(a+b)^3A_1'^2 - (a+b)^2bH + (a+b)b^2R_2' - b^3C_1'\right]\mu'^3 = 0.
\end{aligned}
$$

* The expressions in parentheses carry with them their conjugate expressions.

By substituting proper values in (9) and (10) any invariant of any member of the pencil of conics may be easily found. For instance, by a method to be given presently the discriminant of B* is found to differ only by a numerical factor from the undulation condition. Hence by substitution in (9) and (10) the undulation condition is

$$R_2' - H - A_1'^2 - C_1' = 0, \tag{11}$$

or by reason of (5) and (6)

$$144 C_1' + (B_1' + 2A_1')^2 = 0. \tag{12}$$

Also the invariant whose vanishing is the condition for three concurrent double tangents is the square root of the discriminant of the conic $A - 12B$, or at least it differs only by a factor. Hence the condition for $A - 12B$ to degenerate is

$$-(B_1' + 6A_1')^2 = 0, \tag{13}$$

and for three concurrent double tangents,

$$B_1' + 6A_1' = 0. \tag{14}$$

The conic $A - 16B$ meets R^4 in two points q_i besides the six flexes. By substituting (1) in $A - 16B$ and comparing with the sextic giving the flexes we find a quadratic giving the q's. Its discriminant is

$$B_1' + 8A_1' = 0; \tag{15}$$

hence (15) is the condition for the q's to unite.

Having taken the R^4 in a workable form to find relations among its invariants, we shall now give the most important invariants in their most general forms. It is well known that all line sections of the R^4 are apolar to a pencil of quartics which constitute the *Fundamental Involution*. From the theorem of Grassmann † on the proportionality of determinants it follows that the combinants of the Fundamental Involution are invariants of the R^4. An easy method to pass from the combinants of line sections of an R^4 to combinants of the Fundamental Involution is to replace each determinant by its complementary, i. e., $|acx|$ and $|bkx|$ must be replaced by $|bde|$ and $|ace|$, respectively, etc. Of course, allowance must be made for binomial coefficients. For instance, if these changes are made in the combinant B of Section 1*b* we have

$$\begin{aligned} B' \equiv |bde||abd| - |ade||abe| - |bce||abe| \\ - |acd||ade| + |ace|^2 - |cde||abc| = 0, \end{aligned} \tag{16}$$

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* This was proved in a different way by Dr. THOMSEN in his dissertation, Johns Hopkins University, 1909.

† W. F. MEYER, *Apolarität und Rationale Curven*, § 11.

which is the triple-point condition for R^4 written without binomial coefficients. Further, if these changes are made in A of Section 1*b* the result is -768 times the discriminant of A for R^4 written without binomial coefficients.

We shall summarize the results of these two sections in a table. Let (R^4) mean the R^4 written parametrically with binomial coefficients and $\ulcorner R^4 \urcorner$ mean R^4 written in the same way without them. Also let A' stand for the combinant A of the Fundamental Involution which may be derived from A as already explained. In the first column is given the name of the combinant of two line sections of (R^4); in the second we have the locus resulting from the process explained in Section 1*a*; in the third is the same combinant of the Fundamental Involution giving an invariant of $\ulcorner R^4 \urcorner$ whose meaning appears in the fourth column. Also if A' means an invariant of $\ulcorner R^4 \urcorner$ we shall understand that A'_1 stands for the same invariant of (R^4), possibly differing by a factor. Evidently any such invariant as A'_1 can be obtained from A' by substitution of binomial coefficients, i. e., $|cde|$ and $|bcd|$ of A' would be replaced by $24|cde|$ and $96|bcd|$, respectively, to obtain a multiple of A_1.

Table of Related Invariants and Covariants.

Combinant of two line sections of (R^4).	Covariant Curve of (R^4).	Combinant of F. I. giving an invariant of $\ulcorner R^4 \urcorner$.	Condition for $\ulcorner R^4 \urcorner$ to have
A	Locus whose lines cut (R^4) in self-apolar sets.	A'	3 concurrent flex-tangents.
B	Locus of vertices of flex Δ's of 1st osculants.	B'	Triple point.
C	Product of 4 double tangents.	C'	4 collinear flexes.
D	Product of 6 flex tangents.	D'	Cusp.
R	Point equation of (R^4).	R'	Undulation.
E	Locus whose lines cut (R^4) in catalectic sets.	E'	Tac-node.
$A-12B$	Locus of flex lines of 1st osculants.	$A'-12B'$	Its 2 q's unite.
M	Remote meaning.	M'	Skew invariant.

For the purpose of deriving these invariants a special form of the R^4 has been used; and after they have been derived, to identify them with the proper combinant of the Fundamental Involution is a comparatively easy matter. This was the case with $A'-12B'$, which as a combinant of the Fundamental Involution one would hardly have expected to give the condition for three concurrent

double tangents. On the other hand, forming the cubic of (8), Section $1a$, of the Fundamental Involution, we have three catalectic sets associated with the double points; if two of these are the same we have a tac-node, the condition for which is $E'=0$, and this would probably be a difficult result to obtain directly from the equations of the curve. $D'=0$ is the condition for (R^4) to have a doubly perspective quintic, and therefore the condition for a cusp.

Let the invariants A', B', C', and D' be called I_2, I_2', I_4, and I_6, respectively. *These four invariants constitute a complete system in the sense that any other invariant relation on the R^4 can be expressed in terms of them*, i. e. any other invariant of the R^4 is connected with these four by an algebraic relation. The proof of this fact depends upon a theorem due to Stroh,* which applied here says that every combinant of two binary quartics multiplied by the proper power of an invariant (which in this case is the undulation condition) is an invariant of a binary sextic.† In this case $I_2(R'^2)^2$, $I_2'(R'^2)^2$, $I_4(R'^2)^4$, $I_6(R'^2)^6$, and $I_k(R'^2)^k$ (where k is the degree of an invariant) become J_6, J_6', J_{12}, J_{18}, and J_{3k} of a binary sextic. But I_2, I_2', I_4, and I_6 are independent, hence J_0, J_6, J_{12}, and J_{18} are independent. Suppose then that there is an invariant I_k of the R^4 which gives rise to an I_{3k} of the binary sextic. Then we should have five invariants of a binary sextic, four of which are independent, and these are always connected by an algebraic relation. Consequently any I_k of the R^4 is algebraically expressible in terms of the four invariants I_2, I_2', I_4, and I_6.

§ 3. *The R^4 as Plane Sections of the Steiner Quartic Surface.*

The invariants of the R^4 may be treated very neatly by considering the R^4 as plane sections of the Steiner Quartic Surface ‡ which we shall call S^4.

The point equation of the S^4 referred to its trope planes is

$$(1) \qquad \sqrt{x_i} = \sqrt{x_0} + \sqrt{x_1} + \sqrt{x_2} + \sqrt{x_3} = 0;$$

we propose to cut this by the plane

$$(2) \qquad (\alpha x) = \alpha_0 x_0 + \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 = 0;$$

in this manner any relation among the α's is of (2) yields a special R^4, or the invariants of the R^4 may be expressed in terms of the symmetric functions of the α's of equation (2).

For instance, any plane through the point (1111) cuts out an R^4 with a triple point and therefore the condition for an R^4 with a triple point is that (2) be on

* E. STROH, Mathematische Annalen, vol. 34 (1889), pp 321–323.

† W. STAHL, Crelle's Journal, vol 104 (1889), p. 302.

‡ The Steiner Quartic Surface may be obtained as the polar of a plane as to a tetrahedron. (1) is the polar of the plane (1111) as to $\xi_0\xi_1\xi_2\xi_3=0$. It is of the third class, fourth order, with an enveloping cone of order six. Hence a plane section is a curve of order four and class six—an R^4.

the point (1111), which is $\alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 = 0$, and this as a symmetric function of the α's is

$$S_1 = 0. \tag{3}$$

Any plane on a pinch point cuts out an R^4 with a cusp; the six pinch points have the coordinates (1100), $\cdots$, (0011), hence we have an R^4 with a cusp when any one of the six factors of

$$(\alpha_0 + \alpha_1)(\alpha_0 + \alpha_2)(\alpha_0 + \alpha_3)(\alpha_1 + \alpha_2)(\alpha_1 + \alpha_3)(\alpha_2 + \alpha_3) = 0 \tag{4}$$

vanishes. Expressed in terms of symmetric functions, (4) becomes

$$S_1 S_2 S_3 - S_3^2 - S_1^2 S_4 = 0, \tag{5}$$

which is the general condition for a cusp. Similarly when any two factors of (4) vanish we have two cusps, and by taking the product of the factors of (4) five at a time we obtain the condition for a second cusp after (5) has vanished. I find this to be

$$2S_1 S_2^2 + S_2 S_3 + S_1^2 S_3 - S_1 S_4 = 0. \tag{6}$$

By the same argument, after (5) and (6) have vanished the condition for a third cusp is

$$S_1 S_3 + 32 S_4 + S_2^2 + 2S_1^2 S_2 = 0. \tag{7}$$

If (2) is a tangent plane of (1), the point of tangency is a double point in the curve of the section. Nodes also occur where (2) cuts the double lines of the S^4. The condition for an extra node is that (2) be a plane of (1). The plane equation of (1) is

$$(1/\xi_i) = 1/\xi_0 + 1/\xi_1 + 1/\xi_2 + 1/\xi_3 = 0. \tag{8}$$

and hence the condition for an extra node, or for R^4 to degenerate, is

$$S_3 = 0. \tag{9}$$

The points where (2) cuts the trope conics are points of contact of the double tangents in the curve of section or the double tangent are the intersections of (2) and trope planes. Hence if (2) is on a vertex of the reference tetrahedron the corresponding R^4 has three concurrent double tangents. As this can occur in four ways, the condition for three concurrent double tangents is

$$S_4 = 0. \tag{10}$$

Evidently this could not occur twice unless (9) holds, in which case R_4 degenerates.

The line joining the two points where (2) cuts a trope conic is a double tan-

gent; if these two points come together the intersection of (2) and a trope plane becomes a line of the corresponding trope conic and in the section this point is an undulation. Hence the condition for an undulation is that the intersection of (2) and a trope plane be a line of the corresponding trope conic. This may occur in four ways and the general undulation condition is the product

$$\prod^{4}(\alpha_0\alpha_1 + \alpha_0\alpha_2 + \alpha_1\alpha_2) = 0,$$

or

$$S_2 S_3^2 - S_1 S_3 S_4 + S_4^2 = 0. \tag{11}$$

By the same argument as in case of cusps, the condition for a second undulation after (11) has vanished is found to be

$$S_3^2 + S_1 S_2 S_3 - S_1^2 S_4 + 7 S_2 S_4 = 0. \tag{12}$$

Also for a third, after (11) and (12) have vanished we have

$$S_2^2 + S_1 S_3 + 2 S_4 = 0. \tag{13}$$

By the use of a result of Richmond and Stuart * the quadric associated with the S^4 containing all those points which in plane sections are flexes is found in our notation,

$$8(\alpha_0^3 x_0^2 + \alpha_1^3 x_1^2 + \alpha_2^3 x_2^2 + \alpha_3^3 x_3^2) - S_3(\Sigma x_0^2 - 2\Sigma x_0 x_1) = 0. \tag{14}$$

By its form the quadric in the second parenthesis touches the edges of the tetrahedron at their mid-points, and in sections gives the points of contact of the double tangents. Knowing the meaning of (14) and of the quadric just mentioned and recalling that they (or the corresponding conics in Section 1*b*) are proportional to $A - 16B$ and $48\ B - A$, respectively, we see that the quadric $\Sigma\alpha_0^3 x_0^2 = 0$ is the locus of points which in plane sections yields the conic which is the envelope of flex-lines of all first osculants, as it is proportional to $A - 12B$.

It is desirable to form the discriminant of the pencil of conics as has been done in Section 2. In order to do this in a way that involves the α's symmetrically, we find the plane equation of (14) and substitute the α's for ξ's. Thus by using a result of Salmon † after dividing out by S_3, the discriminant of

$$\mu\Sigma\alpha_0^3 x_0^2 - \lambda S_3(\Sigma x_0^2 - 2\Sigma x_0 x_1) = 0 \tag{15}$$

is exhibited as follows,

$$\begin{aligned} S_4^2\mu^3 - (S_2 S_3^2 - 5 S_1 S_3 S_4 + 8 S_4^2)\mu^2\lambda + 4(S_1^2 S_3^2 - 2 S_2 S_3^2 - S_1 S_3 S_4)\mu\lambda^2 \\ + 4 S_3^2(S_1^2 - 4 S_1)\lambda^3 = 0; \end{aligned} \tag{16}$$

* Proceedings of the London Mathematical Society, series 2, vol. 1 (1904), pp. 129–132.

† *Geometry of Three Dimensions;* p. 50, § 67, p. 58, § 79.

and (16) in the notation of Section 1*b* is the discriminant of

$$(17)\qquad \mu(A-16B)+\lambda(A-48B)=0.$$

Equating (17) to A and substituting in (16) we find the discriminant of A to be

$$(18)\qquad A_2 \equiv [\tfrac{2}{3}(S_1 S_3 - 4S_4)]^2 = 0,$$

therefore three flex tangents are concurrent if

$$(19)\qquad S_1 S_3 - 4S_4 = 0.$$

Or by equating (17) to B we find the discriminant of B

$$(20)\qquad \frac{1}{4.36^2}(S_4^2 + S_2 S_3^2 - S_1 S_3 S_4) = 0,$$

but (20) is only a multiple of (11), showing that B degenerates when R^4 has an undulation.

If we identify (17) with

$$(21)\qquad \nu A + K(A-16B) = 0$$

we find that the discriminant of (21) is

$$(22)\qquad \begin{aligned}&[\tfrac{2}{3}(S_1 S_3 - 4S_4)]^2\nu^3 + \tfrac{4}{27}(S_1^2 S_3^2 - 20 S_1 S_3 S_4 + 64 S_4^2)\nu^2 K\\ &+ \tfrac{4}{81}(64 S_4^2 - 5 S_1^2 S_3^2 + 12 S_2 S_3^2 + 4 S_1 S_2 S_4)\nu K^2\\ &\qquad + \tfrac{4}{81}(8 S_1 S_3 S_4 + S_1^2 S_3^2 - 4 S_2 S_3^2) K^3 = 0.\end{aligned}$$

Let us write equation (22) briefly as

$$(23)\qquad A_2^2\nu^3 + H_2\nu^2 K + R_3 \nu K^2 + C_2 K^3 = 0.$$

The importance of (23) will appear later.

If a point where a double line of S^4 cuts (2) is moved up to be a point of (14) we have brought a node and a flex together and have a flecnode; except that at the two pinch points which are on each double line we get cusps.

The points where the double lines of S^4 cut (2) are

$$(24)\qquad \begin{aligned}x_0 &= x_1 = \alpha_2 + \alpha_3, & x_2 &= x_3 = -(\alpha_0 + \alpha_1);\\ x_0 &= x_2 = \alpha_1 + \alpha_3, & x_1 &= x_3 = -(\alpha_0 + \alpha_2);\\ x_0 &= x_3 = \alpha_1 + \alpha_2, & x_1 &= x_2 = -(\alpha_0 + \alpha_3).\end{aligned}$$

Substituting these in equation (14) and factoring the cusp condition twice out each time, I find that the conditions under which a flecnode can occur are three expressions of the type

$$(25)\qquad (\alpha_0 - \alpha_1)^2(\alpha_2 + \alpha_3) + (\alpha_2 - \alpha_3)^2(\alpha_0 + \alpha_1) = 0;$$

i. e., the vanishing of any one of the three expressions of the type of (25) insures a flecnode.

By arguments similar to those already used the general condition for a flecnode is shown to be

$$(26)\qquad \begin{aligned} 32S_1S_2S_3^2 - 4S_1^2S_2^2S_3 - 64S_3^3 + S_1^4S_2S_3 - 5S_1^3S_3^2 \\ + 16S_1^2S_3S_4 - S_1^5S_4 = 0. \end{aligned}$$

The condition for a second flecnode after (26) is satisfied is

$$(27)\qquad S_1^2S_2^2 - 16S_1S_2S_3 + 48S_3^2 + S_1^3S_3 - 48S_1^2S_4 = 0.$$

Also, for a third, in addition to (26) and (27) we find

$$(28)\qquad S_1S_2 - 6S_3 = 0.$$

Obviously an R^4 cannot have three flecnodes without one of them being a biflecnode. The R^4 is cut in eight points by $A - 16B = 0$; six of these points are flexes and the other two are the points q. For a flecnode, a q and a flex must come together; this could occur only twice and a third flecnode would require two flexes of the R^4 to cross, thus forming a biflecnode.

The matter of biflecnode conditions is somewhat unsatisfactory. To find them we introduce a conic of fundamental importance. The conic

$$(29)\qquad \lambda(x_0^2 + x_1^2 + x_2^2) + 2fx_1x_2 + 2gx_0x_2 + 2hx_0x_1 = 0$$

may be transformed by $x_i = 1/x_i$ into the R^4 referred to its nodal triangle, and its equation is

$$(30)\qquad \lambda(x_1^2x_2^2 + x_0^2x_2^2 + x_0^2x_1^2) + 2fx_0^2x_1x_2 + 2gx_1^2x_0x_2 + 2hx_2^2x_0x_1 = 0.$$

It may be shown that the invariants of the R^4 calculated directly from equation (30) are connected with the invariants we have found from the S^4 by the following relations

$$(31)\qquad \begin{aligned} 2\lambda &= \alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 = S_1, \\ 2f &= \alpha_0 + \alpha_1 - \alpha_2 - \alpha_3, \\ 2g &= \alpha_0 - \alpha_1 + \alpha_2 - \alpha_3, \\ 2h &= \alpha_0 - \alpha_1 - \alpha_2 + \alpha_3. \end{aligned}$$

The equation of a pair of nodal tangents may be written down from (30). If we impose upon each of these tangents the condition that it cut the R^4 in three consecutive points the results are the conditions for a biflecnode. These operations yield three pairs of equations of the type

$$(32)\qquad h\lambda - gf + g\gamma = 0, \qquad gf - h\lambda + g\gamma = 0,$$

where $\lambda = 0$ is the triple-point condition and $\gamma = 0$ is the cusp condition. Ruling out the vanishing of λ or γ, one such pair as (32) will vanish if only any two of the quantities (f, g, h) vanish. A pair of conditions expressible in symmetric functions of the α's which are equivalent to these two conditions are

$$(33) \qquad \sum^{3} f^2 g^2 = 0 \qquad \text{and} \qquad f^2 g^2 h^2 = 0,$$

and these are expressible as

$$(34) \qquad 3S_1^4 - 16S_1^2 S_2 + 16S_2^2 + 16S_1 S_3 - 64S_4 = 0,$$

and

$$(35) \quad S_1^6 - 8S_1^4 S_2 + 16S_1^2 S_2^2 + 48S_2^3 + 16S_1^3 S_3 - 64S_1 S_2 S_3 + 64S_3^2 = 0.$$

It may be easily verified that the conic on the flexes degenerates when (34) and (35) vanish simultaneously.